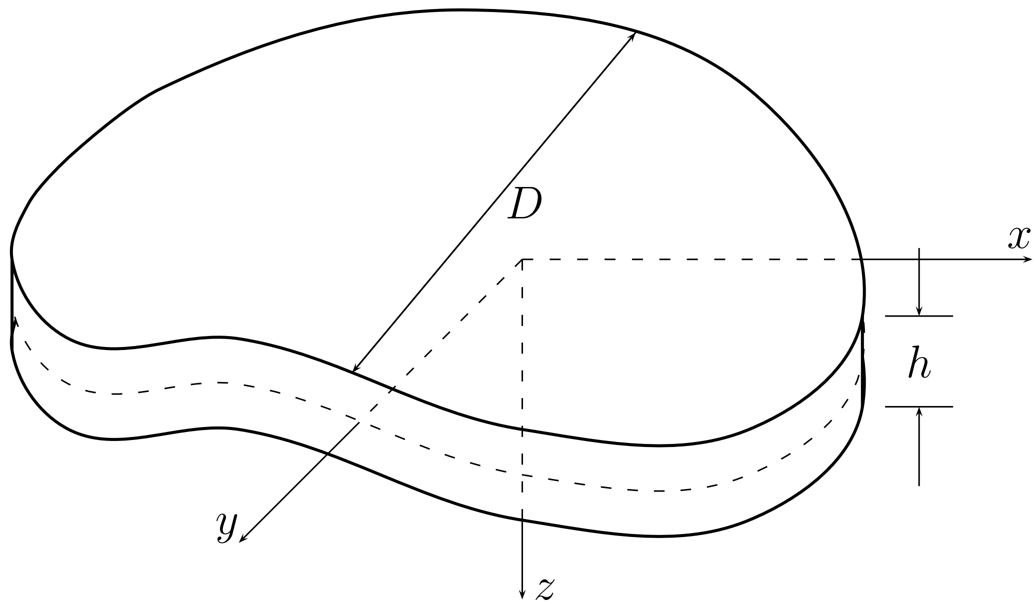
ZGINANIE CIENKIEJ PŁYTY

materiały prezentowane w ramach przedmiotu „wytrzymałość materiałów”

opracował: Z. Więckowski



$$h \ll D$$

Założenia teorii płyt Kirchhoffa

- punkty leżące na normalnej do powierzchni środkowej przed odkształceniem pozostają na normalnej do tej powierzchni po odkształceniu
- naprężenia normalne do powierzchni są pomijalnie małe w porównaniu z pozostałymi składowymi stanu naprężenia

Związki geometryczne

$$\varepsilon_x = \frac{\partial u_x}{\partial x} \tag{1}$$

$$\varepsilon_y = \frac{\partial u_y}{\partial y} \tag{2}$$

$$\varepsilon_z = \frac{\partial u_z}{\partial z} \tag{3}$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \tag{4}$$

$$\gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}$$

$$\gamma_{zx} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}$$

Związki fizyczne

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] \tag{5}$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right] \tag{6}$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right]$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \tag{7}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G}$$

Równania równowagi

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0 \tag{8}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0 \tag{9}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \tag{10}$$

Z założenia geometrycznego, że punkty leżące na normalnej do powierzchni środkowej płyty przed odkształceniem pozostają na normalnej do tej powierzchni po odkształceniu, wynika, że

$$u_x = -\varphi_x\, z$$
$$u_y = -\varphi_y\, z$$

gdzie φ_x i φ_y są kątami obrotu normalnej w płaszczyznach zx i zy. W przypadku małych ugięć płyty prawdziwe są relacje

$$\varphi_x \approx \operatorname{tg}\varphi_x = \frac{\partial w}{\partial x}$$
$$\varphi_y \approx \operatorname{tg}\varphi_y = \frac{\partial w}{\partial y}$$

gdzie $w \equiv w(x, y)$ jest funkcją ugięcia powierzchni środkowej płyty. Zatem przemieszczenia u_x i u_y można wyrazić przez ugięcie powierzchni środkowej płyty następująco:

$$u_x = -\frac{\partial w}{\partial x}\, z$$
$$u_y = -\frac{\partial w}{\partial y}\, z$$

Powyższe wzory oraz związki (1), (2) i (4) prowadzą do następujących wyrażeń na składowe stanu odkształcenia:

$$\varepsilon_x = \frac{\partial u_x}{\partial x} = -\frac{\partial^2 w}{\partial x^2}\, z \tag{11}$$

$$\varepsilon_y = \frac{\partial u_y}{\partial y} = -\frac{\partial^2 w}{\partial y^2}\, z \tag{12}$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = -2\,\frac{\partial^2 w}{\partial x\,\partial y}\, z \tag{13}$$

Założenie 2. ($\sigma_z \ll \sigma_x,\ \sigma_y$) prowadzi do uproszczenia związków fizycznych

$$\varepsilon_x = \frac{1}{E}\,(\sigma_x - \nu\,\sigma_y)$$
$$\varepsilon_y = \frac{1}{E}\,(\sigma_y - \nu\,\sigma_x)$$
$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

które w odwrotnej postaci można przedstawić następująco:

$$\sigma_x = \frac{E}{1-\nu^2}\,(\varepsilon_x + \nu\,\varepsilon_y)$$
$$\sigma_y = \frac{E}{1-\nu^2}\,(\varepsilon_y + \nu\,\varepsilon_x)$$
$$\tau_{xy} = G\,\gamma_{xy}$$

Wykorzystując powyższe związki oraz zależności geometryczne (11)–(13), otrzymujemy

$$\sigma_x = -\frac{E}{1-\nu^2}\,\left(\frac{\partial^2 w}{\partial x^2} + \nu\,\frac{\partial^2 w}{\partial y^2}\right) z \tag{14}$$

$$\sigma_y = -\frac{E}{1-\nu^2}\,\left(\frac{\partial^2 w}{\partial y^2} + \nu\,\frac{\partial^2 w}{\partial x^2}\right) z \tag{15}$$

$$\tau_{xy} = -2\,G\,\frac{\partial^2 w}{\partial x\,\partial y}\,z = -\frac{E}{1+\nu}\,\frac{\partial^2 w}{\partial x\,\partial y}\,z \tag{16}$$

Z równań równowagi (8) i (9) można wyznaczyć naprężenia styczne τ_{zx} i τ_{zy}

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0 \tag{8}$$

$$\Downarrow$$

$$\tau_{zx} = -\int \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} \right) \mathrm{d}z =$$

$$\int \frac{E}{1-\nu^2} \left[\frac{\partial^3 w}{\partial x^3} + \nu \frac{\partial^3 w}{\partial x \, \partial y^2} + (1-\nu) \frac{\partial^3 w}{\partial x \, \partial y^2} \right] z \, \mathrm{d}z =$$

$$\frac{E}{1-\nu^2} \left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \, \partial y^2} \right) \left(\frac{z^2}{2} + C \right)$$

Warunek na wartość naprężenia stycznego na górnej i dolnej powierzchniach płyty

$$\tau_{zx}\big|_{z=\pm\frac{h}{2}} = 0\colon\ \frac{h^2}{8} + C = 0 \Longrightarrow C = -\frac{h^2}{8}$$

$$\tau_{zx} = \frac{E}{1-\nu^2} \left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \, \partial y^2} \right) \left(\frac{z^2}{2} - \frac{h^2}{8} \right)$$

Analogicznie

$$\tau_{zy} = \frac{E}{1-\nu^2} \left(\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x^2 \, \partial y} \right) \left(\frac{z^2}{2} - \frac{h^2}{8} \right)$$

Wykorzystując ostatnie równanie równowagi

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \tag{10}$$

otrzymujemy

$$\begin{aligned}\sigma_z = &- \int \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} \right) \mathrm{d}z = \\ &- \frac{E}{1-\nu^2} \left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{\partial^4 w}{\partial x^2\,\partial y^2} \right) \int \left(\frac{z^2}{2} - \frac{h^2}{8} \right) \mathrm{d}z = \\ &- \frac{E}{1-\nu^2} \left(\frac{\partial^4 w}{\partial x^4} + 2\,\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) \left(\frac{z^3}{6} - \frac{h^2}{8}\,z + B \right)\end{aligned}$$

$$\sigma_z\big|_{z=\frac{h}{2}} = 0\colon\ \ \frac{h^3}{48} - \frac{h^3}{16} + B = 0 \implies B = \frac{h^3}{24}$$

$$\sigma_z = -\frac{E}{1-\nu^2}\,\Delta^2 w \left(\frac{z^3}{6} - \frac{h^2}{8}\,z + \frac{h^3}{24} \right)$$

$$\sigma_z\big|_{z=-\frac{h}{2}} = -q: \quad -\frac{E}{1-\nu^2}\,\Delta^2 w\left(-\frac{h^3}{48}+\frac{h^3}{16}+\frac{h^3}{24}\right) = -q$$

$$\Downarrow$$

równanie powierzchni ugięcia płyty

$$\frac{E\,h^3}{12\,(1-\nu^2)}\,\Delta^2 w = q$$

$$D\,\Delta^2 w = q, \qquad D = \frac{E\,h^3}{12\,(1-\nu^2)}$$

Siły przekrojowe w płycie – momenty zginające, moment skręcający

$$M_x = \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x \, z \, \mathrm{d}z = - \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E}{1-\nu^2} \left(\frac{\partial^2 w}{\partial x^2} + \nu \, \frac{\partial^2 w}{\partial y^2} \right) z^2 \, \mathrm{d}z =$$

$$- \frac{E}{1-\nu^2} \left(\frac{\partial^2 w}{\partial x^2} + \nu \, \frac{\partial^2 w}{\partial y^2} \right) \frac{z^3}{3} \bigg|_{-\frac{h}{2}}^{+\frac{h}{2}} =$$

$$- \frac{E \, h^3}{12 \, (1-\nu^2)} \left(\frac{\partial^2 w}{\partial x^2} + \nu \, \frac{\partial^2 w}{\partial y^2} \right) \equiv D \, (\kappa_x + \nu \, \kappa_y)$$

$$M_y = \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y \, z \, \mathrm{d}z = - \frac{E \, h^3}{12 \, (1-\nu^2)} \left(\frac{\partial^2 w}{\partial y^2} + \nu \, \frac{\partial^2 w}{\partial x^2} \right) \equiv$$

$$D \, (\kappa_y + \nu \, \kappa_x)$$

$$M_{xy} = \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} \, z \, \mathrm{d}z = - \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E}{1+\nu} \frac{\partial^2 w}{\partial x \, \partial y} \, z^2 \, \mathrm{d}z =$$

$$- \frac{E}{1+\nu} \frac{\partial^2 w}{\partial x \, \partial y} \, \frac{z^3}{3} \bigg|_{-\frac{h}{2}}^{+\frac{h}{2}} = - \frac{E \, h^3}{12 \, (1+\nu)} \frac{\partial^2 w}{\partial x \, \partial y} \equiv (1-\nu) \, D \, \kappa_{xy}$$

Siły przekrojowe w płycie – siły poprzeczne

$$Q_x = \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{zx}\,\mathrm{d}z = \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E}{1-\nu^2}\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x\,\partial y^2}\right)\left(\frac{z^2}{2} - \frac{h^2}{8}\right)\mathrm{d}z =$$

$$\frac{E}{1-\nu^2}\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x\,\partial y^2}\right)\left(\frac{z^3}{6} - \frac{h^2}{8}\,z\right)\Bigg|_{-\frac{h}{2}}^{+\frac{h}{2}} =$$

$$-\frac{E\,h^3}{12\,(1-\nu^2)}\,\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \equiv D\,\frac{\partial}{\partial x}\,(\kappa_x + \kappa_y)$$

$$Q_y = \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{zy}\,\mathrm{d}z \equiv D\,\frac{\partial}{\partial y}\,(\kappa_x + \kappa_y)$$

Zależności między siłami przekrojowymi i krzywiznami

$$M_x = D\,(\kappa_x + \nu\,\kappa_y) \qquad \Rightarrow \qquad \kappa_x + \nu\,\kappa_y = \frac{M_x}{D}$$

$$M_y = D\,(\kappa_y + \nu\,\kappa_x) \qquad \Rightarrow \qquad \kappa_y + \nu\,\kappa_x = \frac{M_y}{D}$$

$$M_{xy} = (1-\nu)\,D\,\kappa_{xy} \qquad \Rightarrow \qquad \kappa_{xy} = \frac{M_{xy}}{(1-\nu)\,D}$$

$$Q_x = D\,\frac{\partial}{\partial x}\,(\kappa_x + \kappa_y) \qquad \Rightarrow \qquad \frac{\partial}{\partial x}\,(\kappa_x + \kappa_y) = \frac{Q_x}{D}$$

$$Q_y = D\,\frac{\partial}{\partial y}\,(\kappa_x + \kappa_y) \qquad \Rightarrow \qquad \frac{\partial}{\partial y}\,(\kappa_x + \kappa_y) = \frac{Q_y}{D}$$

Z powyższych związków oraz zależności między naprężeniami i ugięciami płyty wynikają poniższe zależności

$$\sigma_x = \frac{M_x}{h^3/12}\,z$$
$$\sigma_y = \frac{M_y}{h^3/12}\,z$$
$$\tau_{xy} = \frac{M_{xy}}{h^3/12}\,z$$
$$\tau_{zx} = \frac{Q_x}{h^3/12}\left(\frac{h^2}{8} - \frac{z^2}{2}\right)$$
$$\tau_{zy} = \frac{Q_y}{h^3/12}\left(\frac{h^2}{8} - \frac{z^2}{2}\right)$$

Ponadto

$$\sigma_z = -\frac{E}{1-\nu^2}\left(\frac{\partial^4 w}{\partial x^4} + 2\,\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right)\left(\frac{z^3}{6} - \frac{h^2}{8}\,z + \frac{h^3}{24}\right) = -\frac{D}{h^3/12}\,\Delta^2 w\left(\frac{z^3}{6} - \frac{h^2}{8}\,z + \frac{h^3}{24}\right)$$

Z powyższych zależności oraz równań równowagi (10)–(12) wynikają równania równowagi sił wewnętrznych w płycie

$$(10) \Rightarrow \qquad \frac{12\,z}{h^3}\,\frac{\partial M_x}{\partial x} + \frac{12\,z}{h^3}\,\frac{\partial M_{xy}}{\partial y} - \frac{12\,z}{h^3}\,Q_x = 0$$

$$\Downarrow$$

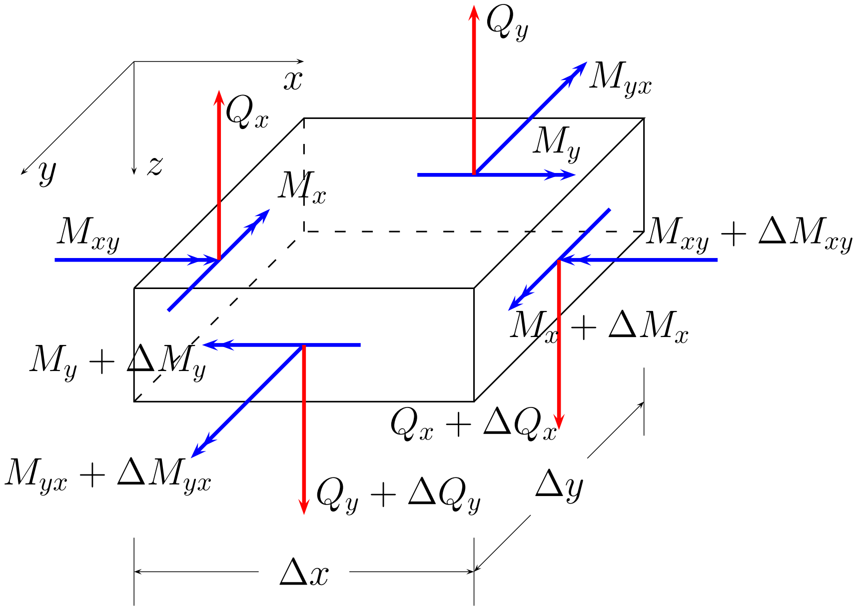
$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0$$

$$(11) \Rightarrow \qquad \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0$$

$$(12) \Rightarrow \qquad \frac{12}{h^3}\left(\frac{h^2}{8} - \frac{z^2}{2}\right)\frac{\partial Q_x}{\partial x} + \frac{12}{h^3}\left(\frac{h^2}{8} - \frac{z^2}{2}\right)\frac{\partial Q_y}{\partial y} - \frac{12}{h^3}\left(\frac{z^2}{2} - \frac{h^2}{8}\right)D\,\Delta^2 w = 0$$

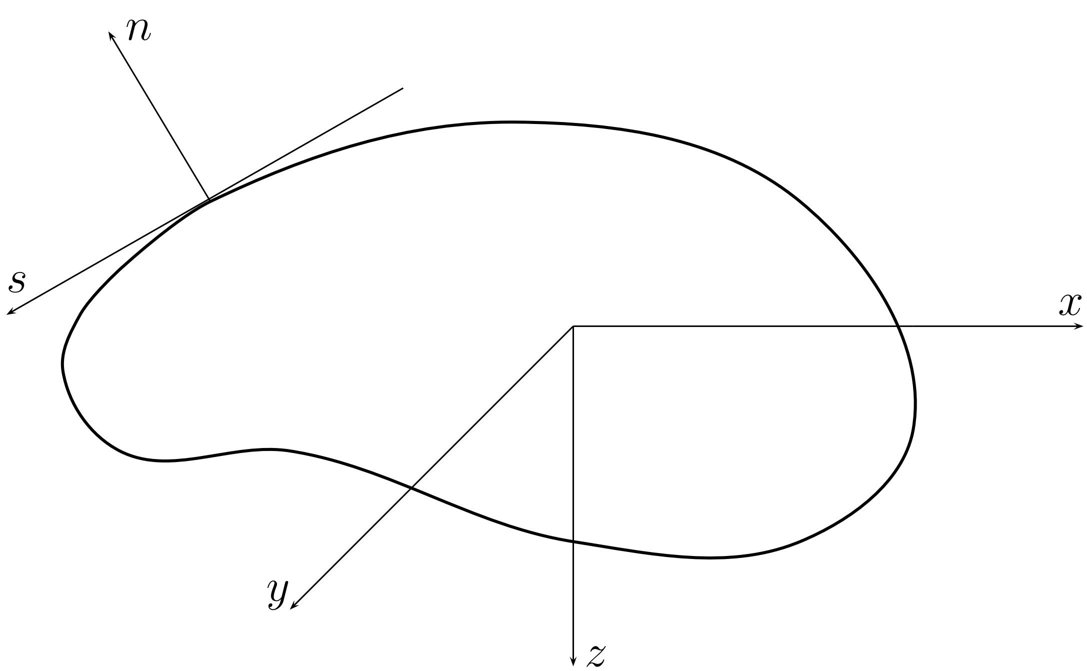
$$\Downarrow$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0$$

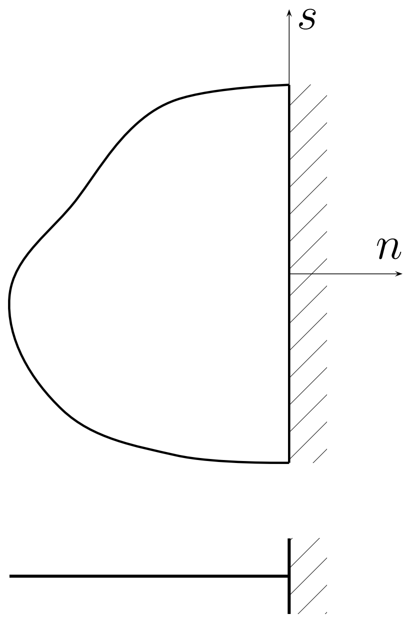


Siły przekrojowe w płycie

Warunki brzegowe



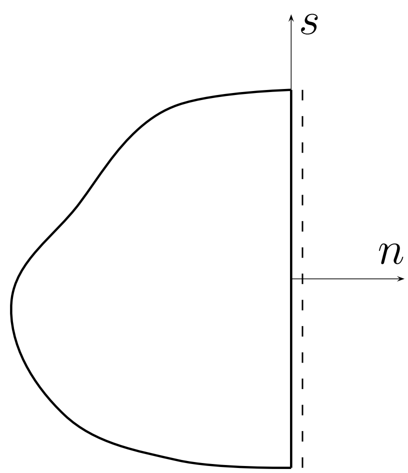
brzeg zamocowany

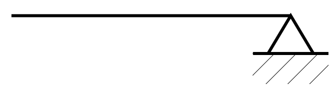


$$w = w_0, \qquad \frac{\partial w}{\partial n} = \varphi_0$$

gdzie $w_0 \equiv w_0(s)$, $\varphi_0 \equiv \varphi_0(s)$ – dane funkcje, w szczególności równe zeru

brzeg swobodnie podparty



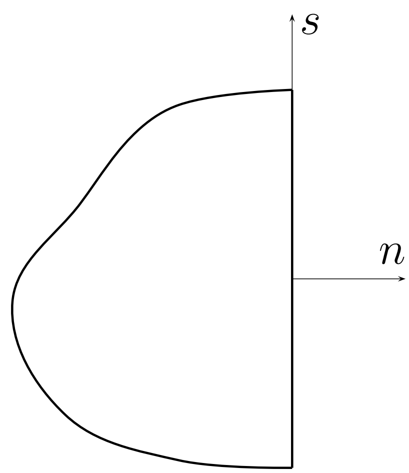


$$w = w_0$$

$$\left.\begin{aligned} \sigma_n = \sigma_n^0 \Rightarrow M_n = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_n^0 \, z \, \mathrm{d}z = M_0 \\ -D\left(\frac{\partial^2 w}{\partial n^2} + \nu \frac{\partial^2 w}{\partial s^2}\right) = M_0 \\ w = 0 \Rightarrow \frac{\partial^2 w}{\partial s^2} = 0 \end{aligned}\right\} \Rightarrow -D\,\frac{\partial^2 w}{\partial n^2} = M_0$$

gdzie $w_0 \equiv w_0(s)$, $M_0 \equiv M_0(s)$ – dane funkcje

brzeg swobodny

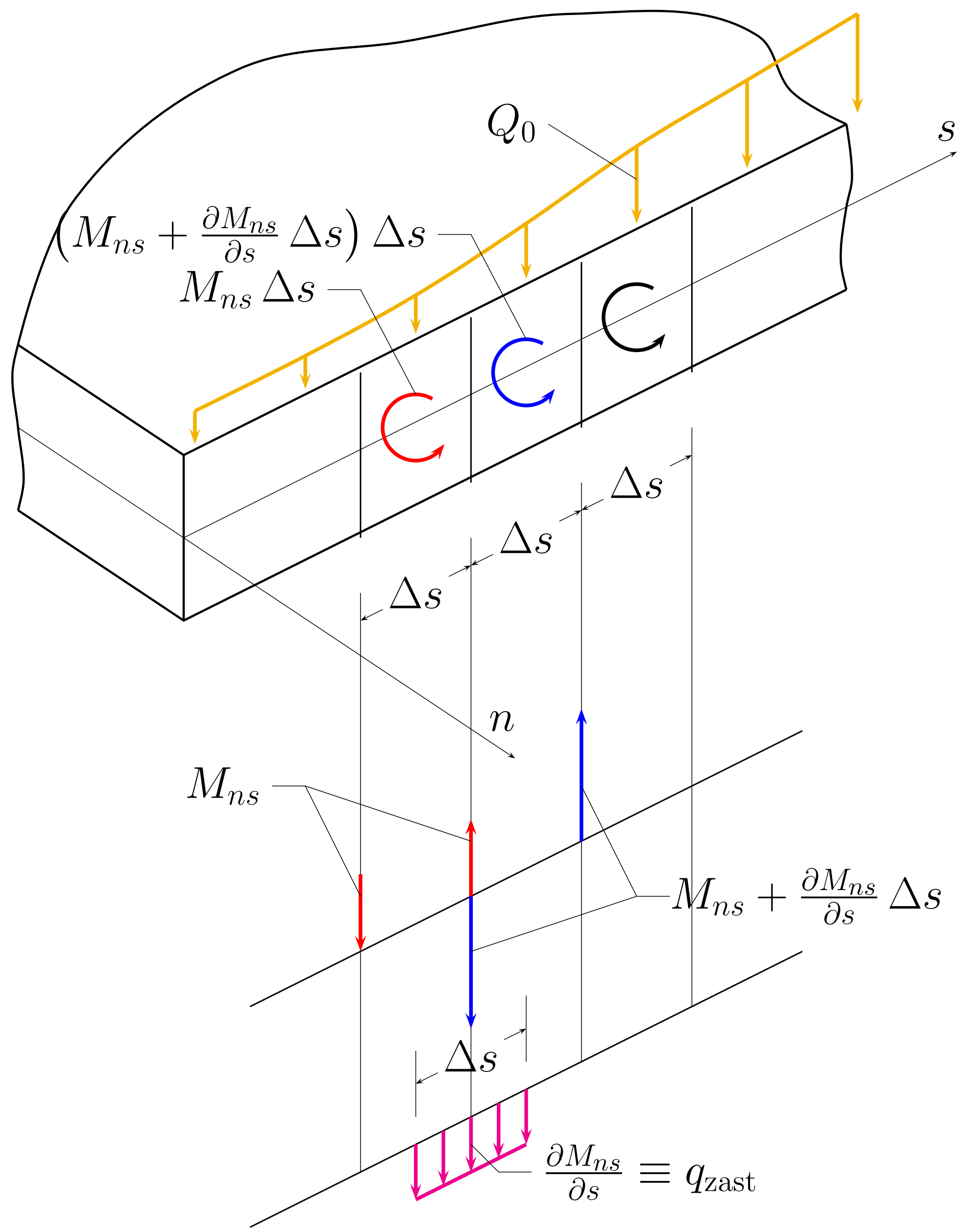


W sformułowaniu 3-wymiarowym trzy warunki naprężeniowe

$$\sigma_n = \sigma_n^0, \qquad \tau_{nz} = \tau_{nz}^0, \qquad \tau_{ns} = \tau_{ns}^0$$

Równanie 4-go rzędu wymaga sformułowania 2 (i tylko 2) warunków brzegowych

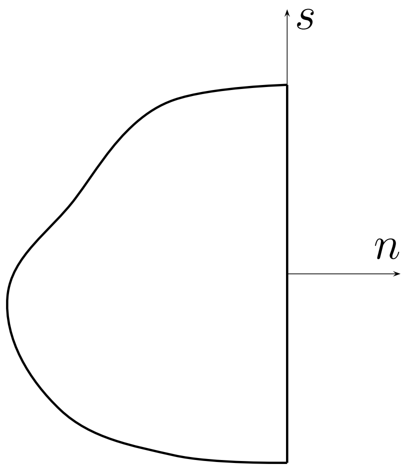
?



Zredukowana siła poprzeczna

$$\overline{Q}_n \equiv Q_n + \frac{\partial M_{ns}}{\partial s} = Q_0$$

brzeg swobodny



$$\sigma_n = \sigma_n^0 \Rightarrow M_n = M_n^0 \Rightarrow -D\left(\frac{\partial^2 w}{\partial n^2} + \nu\,\frac{\partial^2 w}{\partial s^2}\right) = M_0$$

$$\overline{Q}_n \equiv Q_n + \frac{\partial M_{ns}}{\partial s} = Q_0$$

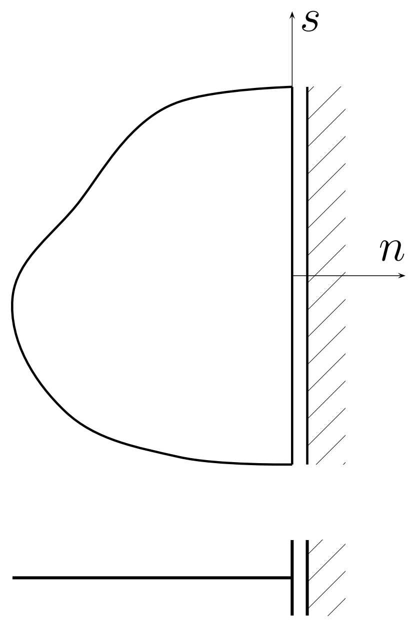
$$\Downarrow$$

$$-D\,\frac{\partial}{\partial n}\left(\frac{\partial^2 w}{\partial n^2} + \frac{\partial^2 w}{\partial s^2}\right) - D\,(1-\nu)\,\frac{\partial}{\partial s}\,\frac{\partial^2 w}{\partial n\,\partial s} = Q_0$$

$$\Downarrow$$

$$-D\left(\frac{\partial^3 w}{\partial n^3} + (2-\nu)\,\frac{\partial^3 w}{\partial n\,\partial s^2}\right) = Q_0$$

oś symetrii



$$\frac{\partial w}{\partial n} = 0 \quad (\varphi_0)$$

$$-D\left(\frac{\partial^3 w}{\partial n^3} + (2-\nu)\,\frac{\partial^3 w}{\partial n\,\partial s^2}\right) = Q_0$$

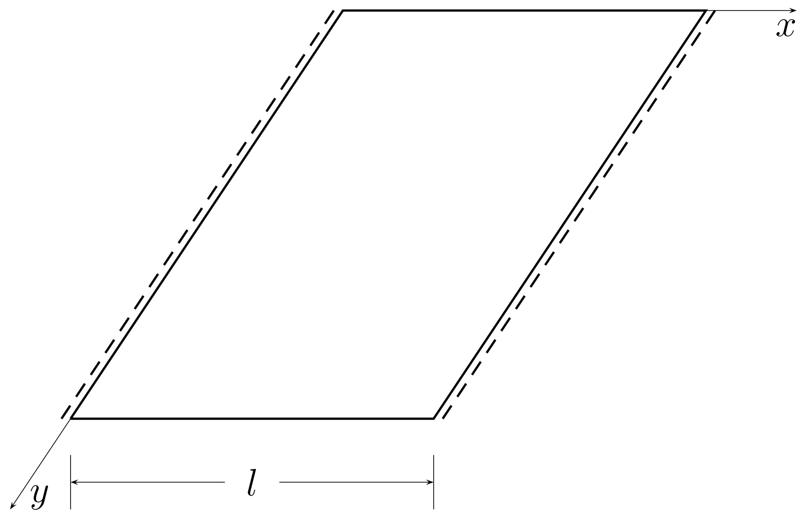
Równanie różniczkowe powierzchni ugięcia płyty

$$\Delta^2 w = \frac{q}{D}$$

$$\Delta^2 w = \Delta\Delta w = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$
$$= \left(\frac{\partial^4 w}{\partial x^4} + 2\,\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right)$$

Zagadnienie brzegowe $=$
równanie różniczkowe $+$ **warunki brzegowe**

Płyta prostokątna podparta swobodnie na dwóch brzegach – rozwiązanie metodą Levy'ego



Funkcja

$$w_n(x,y) = Y_n(y)\,\sin\frac{n\,\pi\,x}{l}$$

spełnia warunki brzegowe na krawędziach $x = 0$ i $x = l$. Zatem funkcja

$$w(x,y) = \sum_{n=1}^{\infty} Y_n(y)\,\sin\frac{n\,\pi\,x}{l}$$

spełnia je również.

$$\frac{\partial^4 w}{\partial x^4} + 2\,\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

$$\frac{\partial^4 w}{\partial x^4} = \sum_{n=1}^{\infty} Y_n(y)\,\frac{n^4\,\pi^4}{l^4}\,\sin\frac{n\,\pi\,x}{l}$$

$$\frac{\partial^4 w}{\partial x^2\,\partial y^2} = -\sum_{n=1}^{\infty} Y_n''(y)\,\frac{n^2\,\pi^2}{l^2}\,\sin\frac{n\,\pi\,x}{l}$$

$$\frac{\partial^4 w}{\partial y^4} = \sum_{n=1}^{\infty} Y_n^{IV}(y)\,\sin\frac{n\,\pi\,x}{l}$$

Rozwinięcie funkcji obciążenia w szereg Fouriera

$$q(x,y) = \sum_{n=1}^{\infty} a_n(y)\,\sin\frac{n\,\pi\,x}{l}$$

$$\sum_{n=1}^{\infty}\left(Y_n^{IV} - 2\,\frac{n^2\,\pi^2}{l^2}\,Y_n'' + \frac{n^4\,\pi^4}{l^4}\,Y_n - \frac{1}{D}\,a_n\right)\sin\frac{n\,\pi\,x}{l} = 0 \quad \forall x$$

$$\Downarrow$$

$$Y_n^{IV} - 2\,\alpha_n^2\,Y_n'' + \alpha_n^4\,Y_n = \frac{1}{D}\,a_n(y), \qquad \alpha_n = \frac{n\,\pi}{l}$$

$$Y_n^{IV} - 2\,\alpha_n^2\,Y_n'' + \alpha_n^4\,Y_n = \frac{1}{D}\,a_n(y), \qquad \alpha_n = \frac{n\,\pi}{l}$$

Rozwiązanie równania jednorodnego poszukujemy w postaci

$$Y_n(y) = e^{r_n y}$$

$$\Downarrow$$

$$r_n^4 - 2\,\alpha_n^2\,r_n^2 + \alpha_n^4 = 0$$

$$\Downarrow$$

$$\left(r_n^2 - \alpha_n^2\right)^2 = [(r_n + \alpha_n)(r_n - \alpha_n)]^2 = 0$$

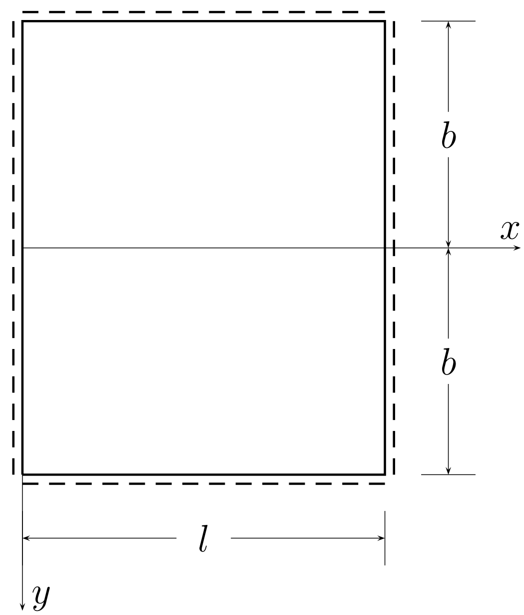
$$r_n = \pm\alpha_n$$

Rozwiązanie ogólne równania niejednorodnego

$$Y_n(y) = A_n\,\mathrm{sh}\,\alpha_n y + B_n\,\mathrm{ch}\,\alpha_n y + C_n\,y\,\mathrm{sh}\,\alpha_n y + D_n\,y\,\mathrm{ch}\,\alpha_n y + \overline{Y}_n$$

gdzie $\overline{Y}_n$ jest rozwiązaniem szczególnym równania niejednorodnego.

Przykład: Płyta swobodnie podparta na czterech bokach, obciążona równomiernie ($q = \mathrm{const}$)



Obliczenie współczynników rozwinięcia funkcji obciążenia

$$q(x,y) = \sum_{n=1}^{\infty} a_n(y)\,\sin\frac{n\,\pi\,x}{l}$$

$$\begin{aligned} q = q(x) \;&\Rightarrow\; a_n = \mathrm{const} \\ q = \mathrm{const} \;&\Rightarrow\; a_n = \mathrm{const} \end{aligned}$$

$$q(x,y) = \sum_{n=1}^{\infty} a_n(y)\,\sin\frac{n\,\pi\,x}{l}\;\Bigg|\quad \cdot\sin\frac{m\,\pi\,x}{l}\,,\quad \int\limits_0^l \mathrm{d}x$$

$$\int\limits_0^l q\,\sin\frac{m\,\pi\,x}{l}\,\mathrm{d}x = \sum_{n=1}^{\infty}\int\limits_0^l a_n\,\sin\frac{n\,\pi\,x}{l}\,\sin\frac{m\,\pi\,x}{l}\,\mathrm{d}x = \int\limits_0^l a_m\,\sin^2\frac{m\,\pi\,x}{l}\,\mathrm{d}x$$

ponieważ

$$\int\limits_0^l \sin\frac{n\,\pi\,x}{l}\,\sin\frac{m\,\pi\,x}{l}\,\mathrm{d}x = 0,\quad \text{jeśli } m \neq n$$

$$a_n = \frac{\int\limits_0^l q(x)\,\sin\frac{n\,\pi\,x}{l}\,\mathrm{d}x}{\int\limits_0^l \sin^2\frac{n\,\pi\,x}{l}\,\mathrm{d}x}$$

$$a_n = q\,\frac{\left[-\frac{l}{n\,\pi}\,\cos\frac{n\,\pi\,x}{l}\right]_0^l}{\int\limits_0^l \frac{1-\cos\frac{2\,n\,\pi\,x}{l}}{2}\,\mathrm{d}x} = -\frac{2\,q\,l}{n\,\pi}\,\frac{\left[\cos\frac{n\,\pi\,x}{l}\right]_0^l}{\left[x-\frac{l}{2\,n\,\pi}\,\sin\frac{2\,n\,\pi\,x}{l}\right]_0^l} =$$

$$= \begin{cases} \dfrac{4\,q}{n\,\pi}, & \text{jeśli } n = 1, 3, 5, \ldots \\ 0, & \text{jeśli } n = 2, 4, 6, \ldots \end{cases}$$

Wyznaczenie rozwiązania szczególnego równania różniczkowego

$$\overline{Y}_n^{IV} - 2\,\alpha_n^2\,\overline{Y}_n'' + \alpha_n^4\,\overline{Y}_n = \frac{1}{D}\,a_n$$

$$\overline{Y}_n = c_n = \mathrm{const}$$

$$\alpha_n^4\,c_n = \frac{1}{D}\,a_n \;\Rightarrow\; c_n = \frac{a_n}{\alpha_n^4\,D}$$

$$c_n = \begin{cases} \dfrac{4\,q}{n\,\alpha_n^4\,\pi\,D}, & \text{jeśli } n = 1, 3, 5, \ldots \\ 0, & \text{jeśli } n = 2, 4, 6, \ldots \end{cases}$$

Rozwiązanie ogólne równania różniczkowego

$$w(x,y) =$$

$$\sum_{n=1}^{\infty} \left(A_n \operatorname{sh} \alpha_n y + B_n \operatorname{ch} \alpha_n y + C_n\, y \operatorname{sh} \alpha_n y + D_n\, y \operatorname{ch} \alpha_n y + c_n\right) \sin \frac{n\,\pi\,x}{l}$$

Układ jest symetryczny względem osi $x \quad \Rightarrow$

funkcja ugięcia powinna być parzysta względem zmiennej y

$$A_n = D_n = 0$$

Ponieważ funkcje $\operatorname{sh}\alpha_n y$ i $y\operatorname{ch}\alpha_n y$ są nieparzyste,

$$w(x,y) = \sum_{n=1}^{\infty} \left(B_n \operatorname{ch}\alpha_n y + C_n\, y \operatorname{sh}\alpha_n y + c_n\right)\, \sin\frac{n\,\pi\,x}{l}$$

Warunki brzegowe

$$w\Big|_{y=\pm b} = 0\colon \quad \sum_{n=1}^{\infty} \left(B_n \operatorname{ch}\alpha_n b + C_n\, b \operatorname{sh}\alpha_n b + c_n\right)\, \sin\frac{n\,\pi\,x}{l} = 0$$

$$\frac{\partial^2 w}{\partial y^2}\Big|_{y=\pm b} = 0\colon \quad \sum_{n=1}^{\infty} \Big[B_n\, \alpha_n^2 \operatorname{ch}\alpha_n b +$$

$$C_n \left(2\,\alpha_n \operatorname{ch}\alpha_n b + \alpha_n^2 b \operatorname{sh}\alpha_n b\right)\Big] \sin\frac{n\,\pi\,x}{l} = 0$$

$$(y \operatorname{sh}\alpha_n y)'' = (\operatorname{sh}\alpha_n y + \alpha_n y \operatorname{ch}\alpha_n y)' = \alpha_n \operatorname{ch}\alpha_n y + \alpha_n \operatorname{ch}\alpha_n y + \alpha_n^2 y \operatorname{sh}\alpha_n y =$$
$$2\,\alpha_n \operatorname{ch}\alpha_n y + \alpha_n^2 y \operatorname{sh}\alpha_n y$$

Otrzymujemy następujący układ liniowych równań algebraicznych na stałe B_n i C_n:

$$\begin{aligned} B_n \operatorname{ch} \alpha_n b + C_n\, b \operatorname{sh} \alpha_n b &= -c_n \\ B_n\, \alpha_n^2 \operatorname{ch} \alpha_n b + C_n \left(2\, \alpha_n \operatorname{ch} \alpha_n b + \alpha_n^2 b \operatorname{sh} \alpha_n b\right) &= 0 \end{aligned}$$

Rozwiązanie układu

$$B_n = C_n = 0, \quad \text{jeśli } n = 2, 4, 6, \ldots$$

$$\left.\begin{aligned} B_n &= -c_n\, \frac{2 \operatorname{ch} \alpha_n b + \alpha_n b \operatorname{sh} \alpha_n b}{2 \operatorname{ch}^2 \alpha_n b} \\ C_n &= c_n\, \frac{\alpha_n}{2 \operatorname{ch} \alpha_n b} \end{aligned}\right\}, \quad \text{jeśli } n = 1, 3, 5, \ldots$$

Funkcja ugięcia

$$w(x,y) = \sum_{n=1,3,5,\dots} \left(-\frac{2\,\mathrm{ch}\,\alpha_n b + \alpha_n b\,\mathrm{sh}\,\alpha_n b}{2\,\mathrm{ch}^2 \alpha_n b}\,\mathrm{ch}\,\alpha_n y + \frac{\alpha_n}{2\,\mathrm{ch}\,\alpha_n b}\,\mathrm{sh}\,\alpha_n y + 1 \right) \cdot$$

$$\cdot \frac{4\,q}{n\,\pi \alpha_n^4\, D}\,\sin \alpha_n x, \qquad \alpha_n = \frac{n\,\pi}{l}$$

Wartość ugięcia w środku płyty kwadratowej ($l = 2b$)

$$w\Big|_{\substack{x=\frac{l}{2}\\ y=0}} = \sum_{n=1,3,5,\dots} \left(-\frac{2\,\mathrm{ch}\,\frac{n\,\pi}{2} + \frac{n\,\pi}{2}\,\mathrm{sh}\,\frac{n\,\pi}{2}}{2\,\mathrm{ch}^2 \frac{n\,\pi}{2}} + 1 \right) \frac{4\,q\,l^4}{(n\,\pi)^5\, D}\,\sin\frac{n\,\pi}{2} =$$

$$= \frac{q\,l^4}{D} \left(\underbrace{\underbrace{0.00410935}_{\Delta=1.21\%} - 0.00005055}_{=\,0.00405880} + \cdots \right) \approx 0.00406\,\frac{q\,l^4}{D}$$

gdzie symbol Δ oznacza błąd względny

Wartość momentu zginającego w środku płyty kwadratowej ($\nu = 0.3$)

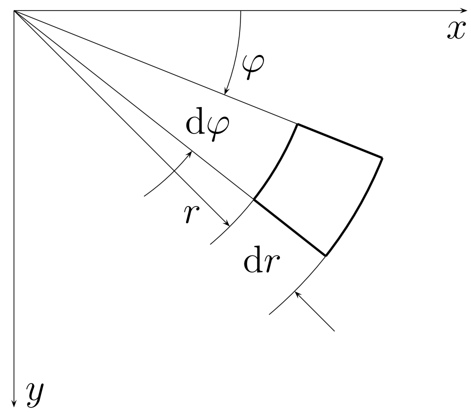
$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\,\frac{\partial^2 w}{\partial y^2}\right)$$

$$M_x\Big|_{\substack{x=\frac{l}{2}\\ y=0}} =$$

$$= \sum_{n=1,3,5,\ldots}\left(-(1-\nu)\,\frac{2\,\mathrm{ch}\,\frac{n\,\pi}{2} + \frac{n\,\pi}{2}\,\mathrm{sh}\,\frac{n\,\pi}{2}}{2\,\mathrm{ch}^2\frac{n\,\pi}{2}} + 1\right)\frac{4\,ql^2}{(n\,\pi)^3}\,\sin\frac{n\,\pi}{2} =$$

$$= ql^2\left(\underbrace{\underbrace{0.051668}_{\Delta=7.9\%} - 0.004551}_{=0.047117,\ \Delta=1.7\ \%} + \cdots\right) \approx 0.0479\,ql^2$$

Płyty kołowe



$$x = r\,\cos\varphi, \qquad y = r\,\sin\varphi$$
$$x^2 + y^2 = r^2, \qquad \operatorname{tg}\varphi = \frac{y}{x}$$
$$r = \sqrt{x^2 + y^2}, \qquad \varphi = \operatorname{arc\,tg}\frac{y}{x}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \varphi}\frac{\partial \varphi}{\partial x} = \cos\varphi\,\frac{\partial f}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial f}{\partial \varphi}$$
$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \varphi}\frac{\partial \varphi}{\partial y} = \sin\varphi\,\frac{\partial f}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial f}{\partial \varphi}$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} = \cos\varphi$$
$$\frac{\partial \varphi}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{y}{r^2} = -\frac{\sin\varphi}{r}$$
$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r} = \sin\varphi$$
$$\frac{\partial \varphi}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \frac{x}{r^2} = \frac{\cos\varphi}{r}$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x^2} = &\left(\cos\varphi\,\frac{\partial}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial}{\partial\varphi}\right)\left(\cos\varphi\,\frac{\partial f}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial f}{\partial\varphi}\right) = \\
&\cos^2\varphi\,\frac{\partial^2 f}{\partial r^2} - \sin\varphi\,\cos\varphi\left(-\frac{1}{r^2}\,\frac{\partial f}{\partial\varphi} + \frac{1}{r}\,\frac{\partial^2 f}{\partial r\,\partial\varphi}\right) - \\
&\frac{\sin\varphi}{r}\left(-\sin\varphi\,\frac{\partial f}{\partial r} + \cos\varphi\,\frac{\partial^2 f}{\partial r\,\partial\varphi}\right) + \frac{\sin\varphi}{r^2}\left(\cos\varphi\,\frac{\partial f}{\partial\varphi} + \sin\varphi\,\frac{\partial^2 f}{\partial\varphi^2}\right) = \\
&\cos^2\varphi\,\frac{\partial^2 f}{\partial r^2} - 2\,\frac{\sin\varphi\,\cos\varphi}{r}\,\frac{\partial^2 f}{\partial r\,\partial\varphi} + 2\,\frac{\sin\varphi\,\cos\varphi}{r^2}\,\frac{\partial f}{\partial\varphi} + \\
&\frac{\sin^2\varphi}{r}\,\frac{\partial f}{\partial r} + \frac{\sin^2\varphi}{r^2}\,\frac{\partial^2 f}{\partial\varphi^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial y^2} = &\left(\sin\varphi\,\frac{\partial}{\partial r} - \frac{\cos\varphi}{r}\frac{\partial}{\partial\varphi}\right)\left(\sin\varphi\,\frac{\partial f}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial f}{\partial\varphi}\right) = \\
&\sin^2\varphi\,\frac{\partial^2 f}{\partial r^2} + \sin\varphi\,\cos\varphi\left(-\frac{1}{r^2}\,\frac{\partial f}{\partial\varphi} + \frac{1}{r}\,\frac{\partial^2 f}{\partial r\,\partial\varphi}\right) + \\
&\frac{\cos\varphi}{r}\left(\cos\varphi\,\frac{\partial f}{\partial r} + \sin\varphi\,\frac{\partial^2 f}{\partial r\,\partial\varphi}\right) + \frac{\cos\varphi}{r^2}\left(-\sin\varphi\,\frac{\partial f}{\partial\varphi} + \cos\varphi\,\frac{\partial^2 f}{\partial\varphi^2}\right) = \\
&\sin^2\varphi\,\frac{\partial^2 f}{\partial r^2} + 2\,\frac{\sin\varphi\,\cos\varphi}{r}\,\frac{\partial^2 f}{\partial r\,\partial\varphi} - 2\,\frac{\sin\varphi\,\cos\varphi}{r^2}\,\frac{\partial f}{\partial\varphi} + \\
&\frac{\cos^2\varphi}{r}\,\frac{\partial f}{\partial r} + \frac{\cos^2\varphi}{r^2}\,\frac{\partial^2 f}{\partial\varphi^2}
\end{aligned}
$$

$$
\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\,\frac{\partial f}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 f}{\partial\varphi^2}
$$

Równanie funkcji ugięcia płyty we współrzędnych biegunowych

$$
\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2}{\partial\varphi^2}\right)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\,\frac{\partial w}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 w}{\partial\varphi^2}\right) = \frac{q}{D}
$$

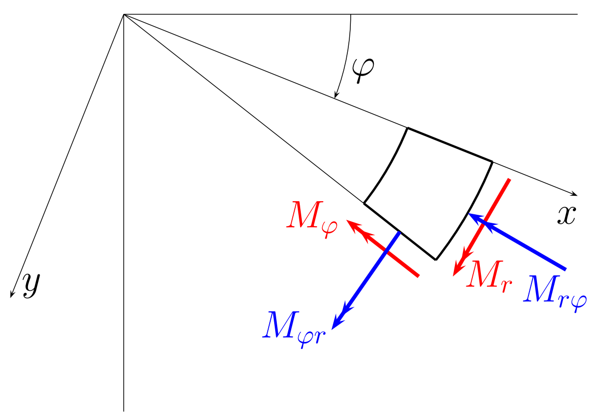
$$\frac{\partial^2 f}{\partial x\,\partial y} = \left(\cos\varphi\,\frac{\partial}{\partial r} - \frac{\sin\varphi}{r}\,\frac{\partial}{\partial\varphi}\right)\left(\sin\varphi\,\frac{\partial f}{\partial r} + \frac{\cos\varphi}{r}\,\frac{\partial f}{\partial\varphi}\right) =$$

$$\sin\varphi\ \cos\varphi\,\frac{\partial^2 f}{\partial r^2} + \cos^2\varphi\left(-\frac{1}{r^2}\,\frac{\partial f}{\partial\varphi} + \frac{1}{r}\,\frac{\partial^2 f}{\partial r\,\partial\varphi}\right) -$$

$$\frac{\sin\varphi}{r}\left(\cos\varphi\,\frac{\partial f}{\partial r} + \sin\varphi\,\frac{\partial^2 f}{\partial r\,\partial\varphi}\right) - \frac{\sin\varphi}{r^2}\left(\sin\varphi\,\frac{\partial f}{\partial\varphi} + \cos\varphi\,\frac{\partial^2 f}{\partial\varphi^2}\right) =$$

$$\sin\varphi\ \cos\varphi\,\frac{\partial^2 f}{\partial r^2} + \frac{\cos 2\varphi}{r}\,\frac{\partial^2 f}{\partial r\,\partial\varphi} - \frac{\cos 2\varphi}{r^2}\,\frac{\partial f}{\partial\varphi} -$$

$$\frac{\sin\varphi\ \cos\varphi}{r}\,\frac{\partial f}{\partial r} - \frac{\sin\varphi\ \cos\varphi}{r^2}\,\frac{\partial^2 f}{\partial\varphi^2}$$



$$M_r = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\,\frac{\partial^2 w}{\partial y^2}\right)\bigg|_{\varphi=0} = -D\left[\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r}\,\frac{\partial w}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 w}{\partial\varphi^2}\right)\right]$$

$$M_\varphi = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\,\frac{\partial^2 w}{\partial x^2}\right)\bigg|_{\varphi=0} = -D\left(\frac{1}{r}\,\frac{\partial w}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 w}{\partial\varphi^2} + \nu\,\frac{\partial^2 w}{\partial r^2}\right)$$

$$M_{r\varphi} = -D\,(1-\nu)\,\frac{\partial^2 w}{\partial x\,\partial y}\bigg|_{\varphi=0} = -D\,(1-\nu)\left(\frac{1}{r}\,\frac{\partial^2 w}{\partial r\,\partial\varphi} - \frac{1}{r^2}\,\frac{\partial w}{\partial\varphi}\right)$$

$$Q_r = -D\,\frac{\partial}{\partial x}\,\Delta w\bigg|_{\varphi=0} = -D\frac{\partial}{\partial r}(\Delta w)$$

$$Q_\varphi = -D\,\frac{\partial}{\partial y}\,\Delta w\bigg|_{\varphi=0} = -D\frac{1}{r}\,\frac{\partial}{\partial \varphi}(\Delta w)$$

Warunki brzegowe

brzeg zamocowany

$$w|_{r=a} = w_0, \qquad \frac{\partial w}{\partial r}\bigg|_{r=a} = \varphi_0$$

brzeg swobodnie podparty

$$w|_{r=a} = w_0, \qquad M_r|_{r=a} = M_0$$

brzeg swobodny

$$M_r|_{r=a} = M_0, \qquad \left(Q_r - \frac{1}{r}\,\frac{\partial M_{r\varphi}}{\partial \varphi}\right)\bigg|_{r=a} = Q_0$$

Rozwiązanie równania płyty we współrzędnych biegunowych

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2}\right)=\frac{q}{D}$$

$$w(r,\varphi)=R_0(r)+\sum_{n=1}^{\infty}\left(R_{cn}(r)\cos n\varphi+R_{sn}(r)\sin n\varphi\right)$$

Osiowo symetryczny przypadek zginania płyty

$$w(r,\varphi) \equiv w(r) \ \Rightarrow \ \frac{\partial w}{\partial \varphi} = 0, \qquad M_{r\varphi} = 0$$

$$\Downarrow$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}\right)\left(\frac{\mathrm{d}^2 w}{\mathrm{d}r^2} + \frac{1}{r}\,\frac{\mathrm{d}w}{\mathrm{d}r}\right) = \frac{q(r)}{D}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r} = \frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r\,\frac{\mathrm{d}}{\mathrm{d}r}\right)$$

$$\Downarrow$$

$$\Delta^2 w = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left\{r\,\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r\,\frac{\mathrm{d}w}{\mathrm{d}r}\right)\right]\right\} = \frac{q}{D}$$

Rozwiązanie równania w przypadku $q = \mathrm{const}$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r}\left\{r\,\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r\,\frac{\mathrm{d}w}{\mathrm{d}r}\right)\right]\right\} &= \frac{q}{D}\,r \\
\left\{r\,\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r\,\frac{\mathrm{d}w}{\mathrm{d}r}\right)\right]\right\} &= \frac{q}{D}\,\frac{r^2}{2} + a \\
\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r\,\frac{\mathrm{d}w}{\mathrm{d}r}\right)\right] &= \frac{q}{D}\,\frac{r}{2} + \frac{a}{r} \qquad (*) \\
\left[\frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r\,\frac{\mathrm{d}w}{\mathrm{d}r}\right)\right] &= \frac{q}{D}\,\frac{r^2}{4} + a\ln r + b \\
\frac{\mathrm{d}}{\mathrm{d}r}\left(r\,\frac{\mathrm{d}w}{\mathrm{d}r}\right) &= \frac{q}{D}\,\frac{r^3}{4} + a\,r\,\ln r + b\,r \\
\left(r\,\frac{\mathrm{d}w}{\mathrm{d}r}\right) &= \frac{q}{D}\,\frac{r^4}{16} + a\,\frac{1}{4}\,r^2\,(2\ln r - 1) + \frac{1}{2}\,b\,r^2 + c = \\
&= \frac{q}{D}\,\frac{r^4}{16} + a'\,r^2\,\ln r + b'\,r^2 + c \\
\frac{\mathrm{d}w}{\mathrm{d}r} &= \frac{q}{D}\,\frac{r^3}{16} + a'\,r\ln r + b'\,r + \frac{c}{r} \\
w &= \frac{q}{D}\,\frac{r^4}{64} + a'\,\frac{1}{4}\,r^2\,(2\ln r - 1) + \frac{1}{2}\,b'\,r^2 + c\ln r + d = \\
&= \frac{q}{D}\,\frac{r^4}{64} + C_1\,r^2\ln r + C_2\,r^2 + C_3\ln r + C_4
\end{aligned}
$$

Płyta kołowa bez otworu (4 stałe, 2 warunki brzegowe)

$$
\left.\begin{aligned}
\lim_{r\to 0}\ln r = -\infty \\
|w|_{r=0} < \infty
\end{aligned}\right\} \;\Rightarrow\; C_3 = 0
$$

$$
Q_r \equiv -D\,\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r\,\frac{\mathrm{d}w}{\mathrm{d}r}\right)\right] = -\frac{\pi r^2 q}{2\pi r} = -\frac{1}{2}\,qr \;\Rightarrow\; C_1 = 0
$$

Rozwiązanie w przypadku dowolnego obciążenia $q(r)$

$$w = \frac{\bar{q}(r)}{D} + C_1\, r^2 \ln r + C_2\, r^2 + C_3 \ln r + C_4$$

gdzie $\bar{q}(r)$ – rozwiązanie szczególne równania niejednorodnego

Przykład: płyta kołowa o promieniu a obciążona obciążeniem stałym na całej powierzchni

rozwiązanie ogólne

$$w = \frac{q}{D}\,\frac{r^4}{64} + C_2\,r^2 + C_4$$

warunki brzegowe

$$\begin{aligned} w(a) = 0: &\qquad \frac{q}{D}\,\frac{a^4}{64} + C_2\,a^2 + C_4 = 0 \\ \frac{\mathrm{d}w}{\mathrm{d}r}(a) = 0: &\qquad \frac{q}{D}\,\frac{a^3}{16} + 2\,C_2\,a = 0 \end{aligned}$$

$$\begin{aligned} C_2 &= -\frac{1}{32}\frac{q}{D}\,a^2 \\ C_4 &= \frac{1}{64}\frac{q}{D}\,a^4 \end{aligned}$$

$$\begin{aligned} w(r) &= \frac{1}{64}\,\frac{q}{D}\left(r^4 - 2\,a^2\,r^2 + a^4\right) = \frac{1}{64}\,\frac{q}{D}\left(a^2 - r^2\right)^2 = \\ &= \frac{1}{64}\,\frac{qa^4}{D}\left[1 - \left(\frac{r}{a}\right)^2\right]^2 \end{aligned}$$

Obliczenie momentów zginających ($M_{r\varphi} = 0$)

$$\frac{\mathrm{d}w}{\mathrm{d}r} = \frac{1}{64}\,\frac{q}{D} \cdot 2\,(a^2 - r^2)\,(-2\,r) = -\frac{1}{16}\,\frac{q}{D}\,r\,(a^2 - r^2)$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}r^2} = -\frac{1}{16}\,\frac{q}{D}\,(a^2 - 3\,r^2)$$

$$M_r = -D\left(\frac{\mathrm{d}^2w}{\mathrm{d}r^2} + \nu\,\frac{1}{r}\,\frac{\mathrm{d}w}{\mathrm{d}r}\right) = \frac{qa^2}{16}\left\{1 - 3\left(\frac{r}{a}\right)^2 + \nu\left[1 - \left(\frac{r}{a}\right)^2\right]\right\}$$

$$M_\varphi = -D\left(\frac{1}{r}\,\frac{\mathrm{d}w}{\mathrm{d}r} + \nu\,\frac{\mathrm{d}^2w}{\mathrm{d}r^2}\right) = \frac{qa^2}{16}\left\{1 - \left(\frac{r}{a}\right)^2 + \nu\left[1 - 3\left(\frac{r}{a}\right)^2\right]\right\}$$

$$M_r = \frac{qa^2}{16}\left[1 + \nu - (3 + \nu)\,\left(\frac{r}{a}\right)^2\right]$$

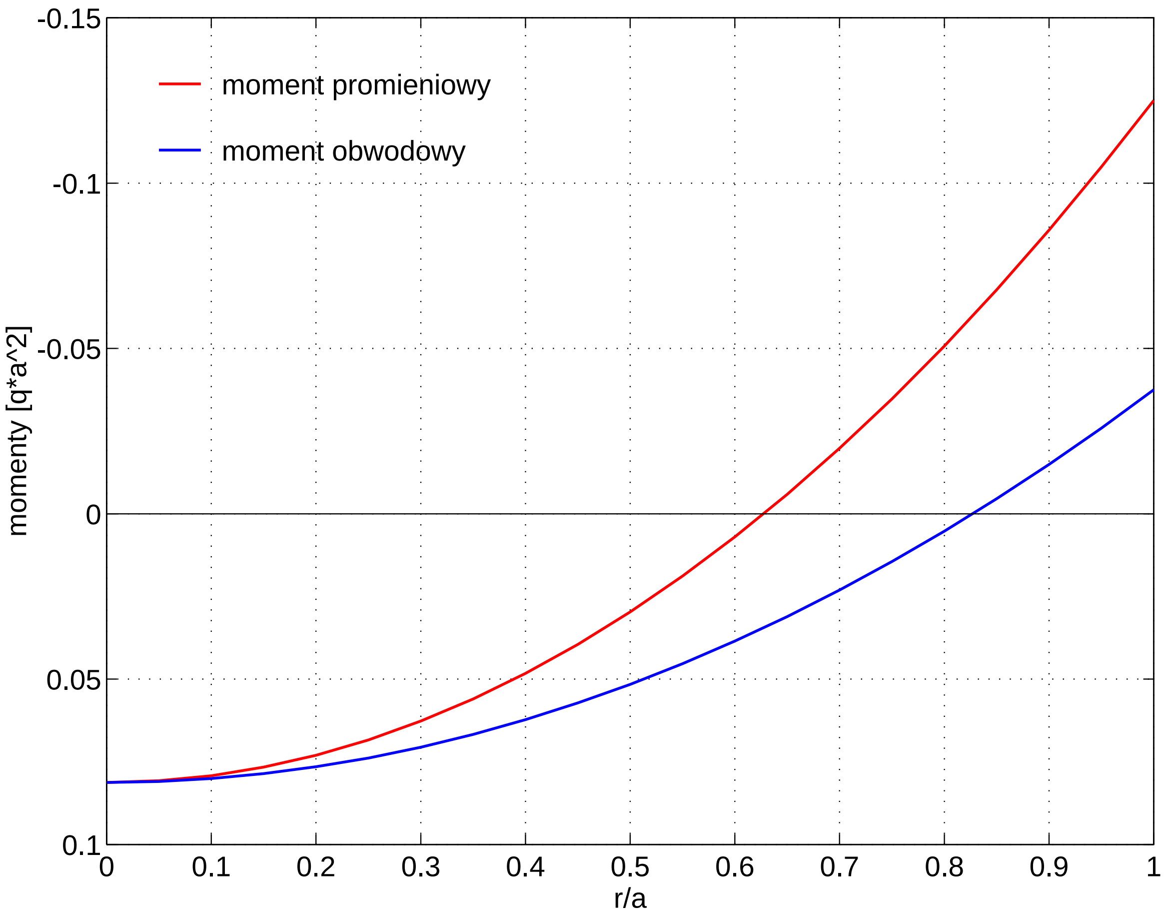
$$M_\varphi = \frac{qa^2}{16}\left[1 + \nu - (1 + 3\,\nu)\,\left(\frac{r}{a}\right)^2\right]$$

$$M_r^{\max} = M_r(0) = \frac{1+\nu}{16}\, qa^2$$

$$M_\varphi^{\max} = M_\varphi(0) = \frac{1+\nu}{16}\, qa^2$$

$$M_r^{\min} = M_r(a) = \frac{1}{8}\, qa^2$$

$$M_\varphi^{\min} = M_\varphi(a) = \frac{1}{8}\, \nu\, qa^2$$



Wykresy momentów: M_r, M_φ ($\nu = 0.3$)

Przykład: płyta kołowa o promieniu a obciążona siłą skupioną P działającą w środku płyty

Wyznaczenie siły poprzecznej z warunku równowagi ($\sum P_{iz}$) odniesionego do fragmentu płyty o promieniu r

$$Q_r(r) = -D\,\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r\,\frac{\mathrm{d}w}{\mathrm{d}r}\right)\right] = -\frac{P}{2\,\pi r}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r\,\frac{\mathrm{d}w}{\mathrm{d}r}\right)\right] &= \frac{P}{2\,\pi D}\,\frac{1}{r}\\
\frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r\,\frac{\mathrm{d}w}{\mathrm{d}r}\right) &= \frac{P}{2\,\pi D}\ln r + A\\
\frac{\mathrm{d}}{\mathrm{d}r}\left(r\,\frac{\mathrm{d}w}{\mathrm{d}r}\right) &= \frac{P}{2\,\pi D}\,r\ln r + A\,r\\
r\,\frac{\mathrm{d}w}{\mathrm{d}r} &= \frac{P}{2\,\pi D}\,\frac{1}{4}\,r^2\,(2\ln r - 1) + \frac{1}{2}\,A\,r^2 + B\\
\frac{\mathrm{d}w}{\mathrm{d}r} &= \frac{P}{2\,\pi D}\,\frac{1}{4}\,r\,(2\ln r - 1) + \frac{1}{2}\,A\,r + B\,\frac{1}{r}\\
w &= \frac{P}{8\,\pi D}\,(r^2\ln r - r^2) + \frac{1}{4}\,A\,r^2 + +B\ln r + C
\end{aligned}$$

Warunek ograniczoności przemieszczeń

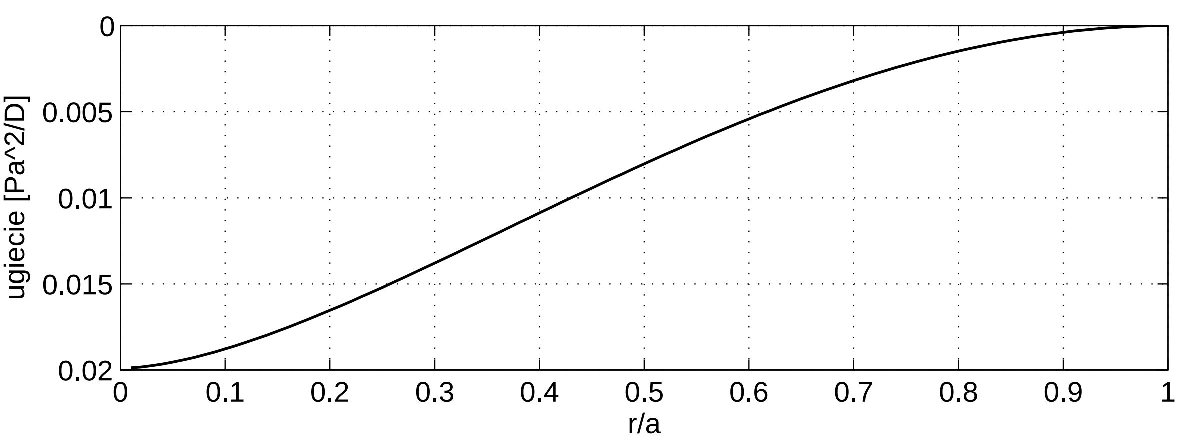
$$\left.\begin{aligned}\lim_{r\to 0}\ln r &= -\infty\\ |w|_{r=0} &< \infty\end{aligned}\right\} \Rightarrow\ B = 0$$

Warunki brzegowe

$$w(a) = 0: \qquad \frac{P}{8\pi D}\, a^2(\ln a - 1) + \frac{1}{4}\, A\, a^2 + C = 0$$
$$\frac{\mathrm{d}w}{\mathrm{d}r}(a) = 0: \qquad \frac{P}{8\pi D}\, a\,(2\ln a - 1) + \frac{1}{2}\, A\, a = 0$$

$$A = -\frac{P}{4\pi D}\,(2\ln a - 1)$$
$$C = \frac{P}{16\pi D}\, a^2$$

$$w = \frac{Pa^2}{16\pi D}\left[2\left(\frac{r}{a}\right)^2 \ln\frac{r}{a} + 1 - \left(\frac{r}{a}\right)^2\right]$$
$$w_{\max} \equiv w(0) = \frac{Pa^2}{16\pi D}$$



Wykres funkcji ugięcia ($\nu = 0.3$)

Obliczenie momentów zginających ($M_{r\varphi} = 0$)

$$\frac{\mathrm{d}w}{\mathrm{d}r} = \frac{P}{16\pi D}\left(4\,r\ln\frac{r}{a} + 2\,r^2\,\frac{a}{r}\,\frac{1}{a} - 2\,r\right) = \frac{P}{4\pi D}\,r\ln\frac{r}{a}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}r^2} = \frac{P}{4\pi D}\left(\ln\frac{r}{a} + r\,\frac{a}{r}\,\frac{1}{a}\right) = \frac{P}{4\pi D}\left(\ln\frac{r}{a} + 1\right)$$

$$M_r = -D\left(\frac{\mathrm{d}^2w}{\mathrm{d}r^2} + \nu\,\frac{1}{r}\,\frac{\mathrm{d}w}{\mathrm{d}r}\right) = -\frac{P}{4\pi}\left(\ln\frac{r}{a} + 1 + \nu\,\ln\frac{r}{a}\right)$$

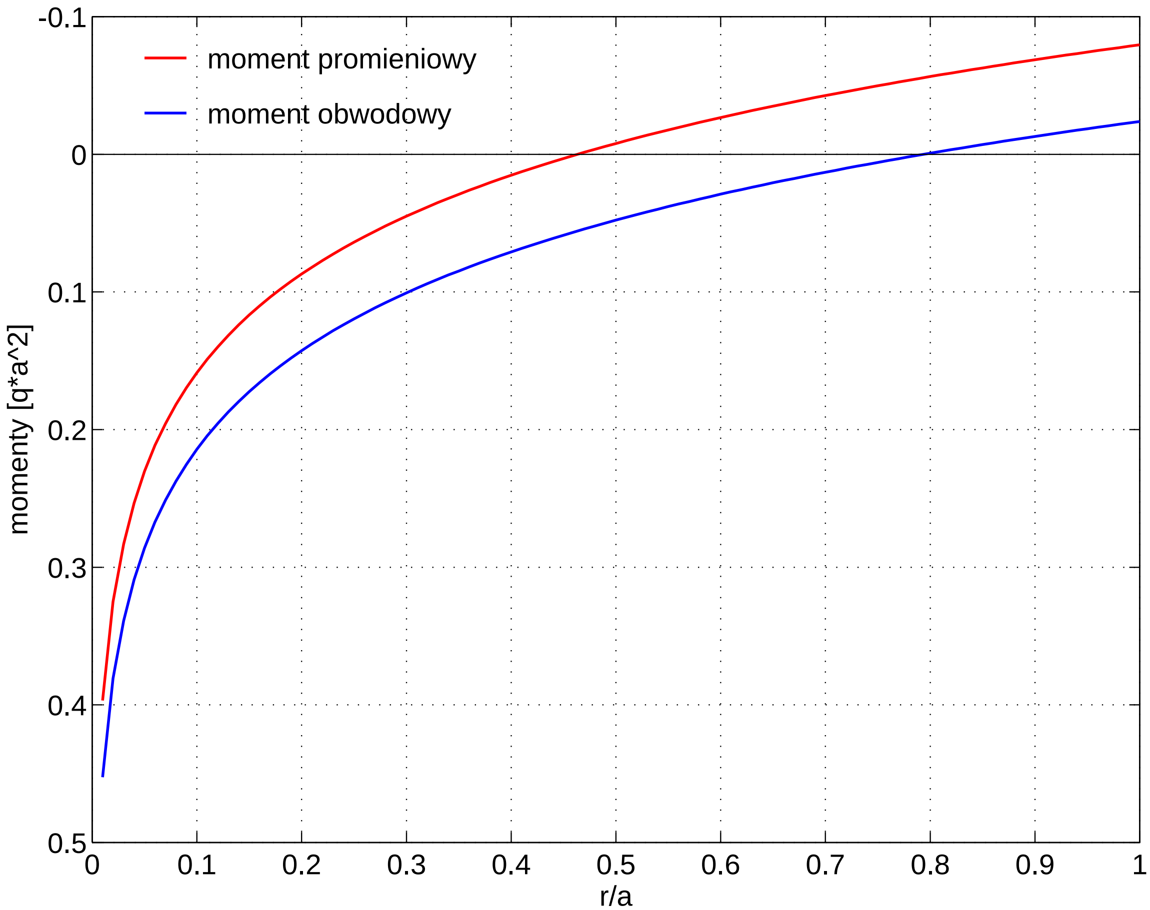
$$M_\varphi = -D\left(\frac{1}{r}\,\frac{\mathrm{d}w}{\mathrm{d}r} + \nu\,\frac{\mathrm{d}^2w}{\mathrm{d}r^2}\right) = -\frac{P}{4\pi}\left(\ln\frac{r}{a} + \nu\,\ln\frac{r}{a} + \nu\right)$$

$$M_r = -\frac{P}{4\pi}\left[(1+\nu)\ln\frac{r}{a} + 1\right]$$

$$M_\varphi = -\frac{P}{4\pi}\left[(1+\nu)\ln\frac{r}{a} + \nu\right]$$

$$M_r = -\frac{P}{4\pi}\left[(1+\nu)\ln\frac{r}{a} + 1\right]$$

$$M_\varphi = -\frac{P}{4\pi}\left[(1+\nu)\ln\frac{r}{a} + \nu\right]$$



Wykresy momentów: M_r, M_φ ($\nu = 0.3$)