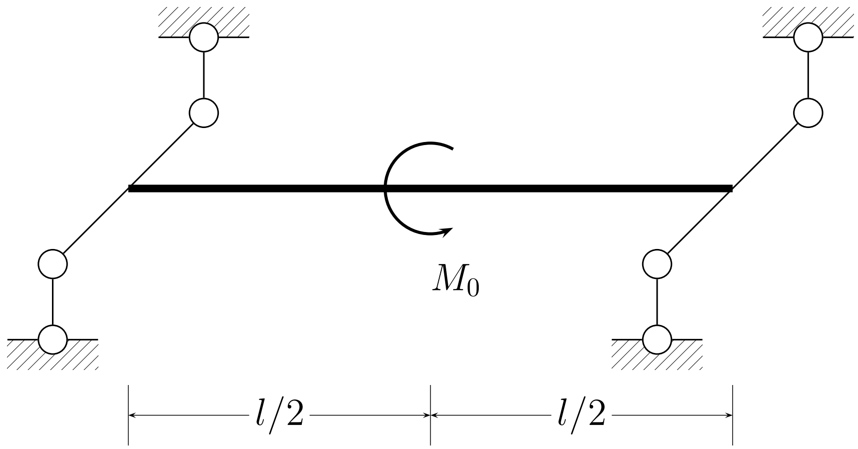
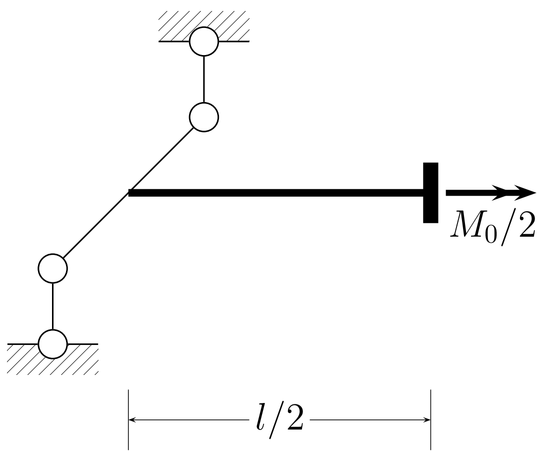
Przykład 1.





rozwiązanie ogólne równania różniczkowego

$$\varphi(x) = A\,\mathrm{sh}kx + B\,\mathrm{ch}kx + C\,x + D$$

pochodne funkcji kąta skręcenia

$$\varphi'(x) = Ak\,\mathrm{ch}kx + Bk\,\mathrm{sh}kx + C$$
$$\varphi''(x) = Ak^2\,\mathrm{sh}kx + Bk^2\,\mathrm{ch}kx$$
$$\varphi'''(x) = Ak^3\,\mathrm{ch}kx + Bk^3\,\mathrm{sh}kx$$

warunki brzegowe

$$\varphi(0) = 0 \quad \Rightarrow \quad B + D = 0 \quad \Rightarrow D = 0$$
$$\varphi''(0) = 0 \quad \Rightarrow \quad B\,k^2 = 0 \quad \Rightarrow B = 0$$
$$\varphi'(\frac{l}{2}) = 0 \quad \Rightarrow \quad A\,k\,\mathrm{ch}\frac{kl}{2} + C = 0$$
$$-EJ_\omega\,\varphi'''(\frac{l}{2}) + GJ_s\,\varphi'(\frac{l}{2}) = \frac{M_0}{2} \quad \Rightarrow \quad -EJ_\omega\,A\,k^3\,\mathrm{ch}k\frac{l}{2} = \frac{M_0}{2}$$

$$A = -\frac{M_0}{2\,EJ_\omega\,k^3\,\mathrm{ch}\frac{kl}{2}}$$
$$C = \frac{M_0}{2\,GJ_s}$$

$$\varphi(x) = \frac{M_0}{2\,k\,GJ_s}\left(kx - \frac{\mathrm{sh}kx}{\mathrm{ch}\frac{kl}{2}}\right)$$

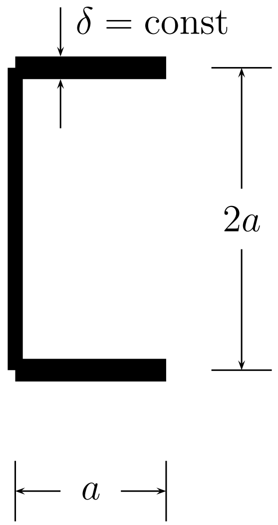
$$\varphi'(x) = \frac{M_0}{2\,GJ_s}\left(1 - \frac{\mathrm{ch}kx}{\mathrm{ch}\frac{kl}{2}}\right) \qquad M_V = GJ_s\,\varphi' = \frac{M_0}{2}\left(1 - \frac{\mathrm{ch}kx}{\mathrm{ch}\frac{kl}{2}}\right)$$

$$\varphi''(x) = -\frac{M_0}{2\,GJ_s}\frac{k\,\mathrm{sh}kx}{\mathrm{ch}\frac{kl}{2}} \qquad B_\omega = -EJ_\omega\,\varphi'' = \frac{M_0}{2\,k}\,\frac{\mathrm{sh}kx}{\mathrm{ch}\frac{kl}{2}}$$

$$\varphi'''(x) = -\frac{M_0}{2\,GJ_s}\frac{k^2\,\mathrm{ch}kx}{\mathrm{ch}\frac{kl}{2}} \qquad M_\omega = -EJ_\omega\,\varphi''' = \frac{M_0}{2}\,\frac{\mathrm{ch}kx}{\mathrm{ch}\frac{kl}{2}}$$

$$M_s = M_\omega + M_V = \frac{M_0}{2}$$

Dane:



$$M_0 = 5 \text{ Nm}$$

$$l = 1 \text{ m}$$

$$a = 4 \text{ cm}$$

$$\delta = \left\{ \begin{matrix} 5 \\ 1 \end{matrix} \right\} \text{ mm}$$

$$E = 2.09 \cdot 10^{11} \text{ Pa}, \;\; \nu = 0.3 \quad \text{(stal)}$$

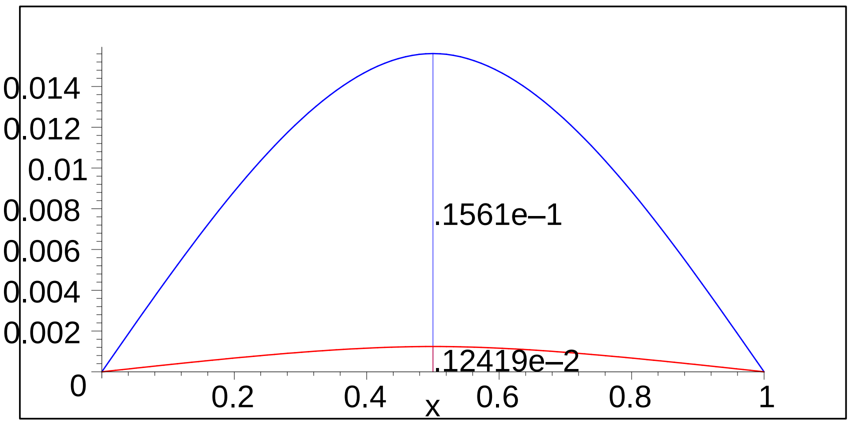
$$J_\omega = \frac{7}{24}\,\delta a^5, \qquad J_s = \frac{1}{3}\,\delta^3\,(a + a + 2a) = \frac{4}{3}\,a\delta^3$$

$$k = \sqrt{\frac{GJ_s}{EJ_\omega}} = \sqrt{\frac{E}{2\,(1+\nu)\,E}\,\frac{\frac{4}{3}\,a\delta^3}{\frac{7}{24}\,\delta a^5}} = \sqrt{\frac{16}{7\,(1+\nu)}\,\frac{\delta}{a^2}}$$

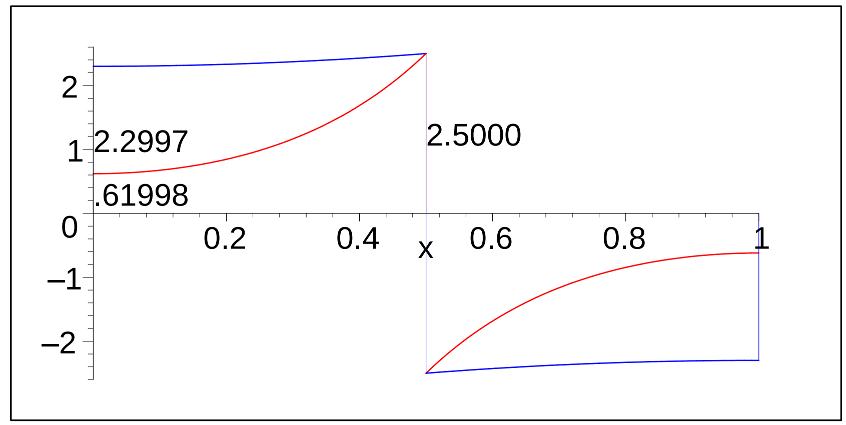
$$\delta = 5\text{mm}: \qquad k = \sqrt{\frac{16}{7\,(1+0.3)}\,\frac{0.5}{4^2}} \qquad = 4.14 \cdot 10^{-2}\,\frac{1}{\text{cm}} = 4.14\,\frac{1}{\text{m}}$$

$$\delta = 1\text{mm}: \qquad k = \sqrt{\frac{16}{7\,(1+0.)}\,\frac{0.1}{4^2}} \qquad = 8.29 \cdot 10^{-3}\,\frac{1}{\text{cm}} = 0.829\,\frac{1}{\text{m}}$$

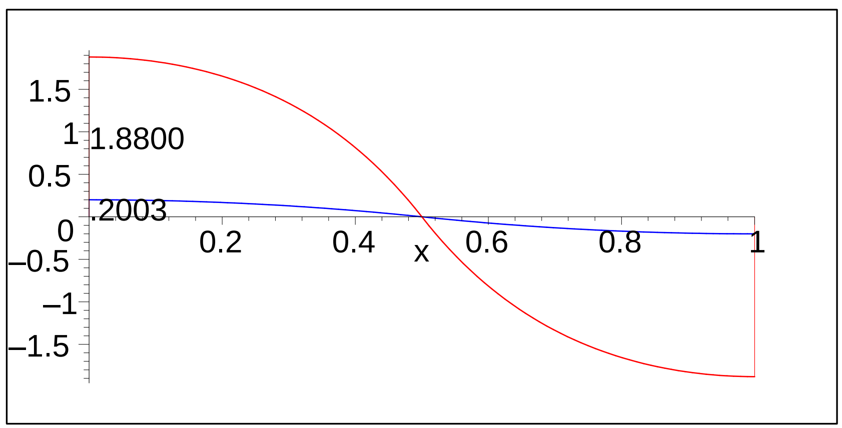
$k = 4.14 \ \frac{1}{\mathrm{m}}$ $\qquad$ $k = 0.829 \ \frac{1}{\mathrm{m}}$



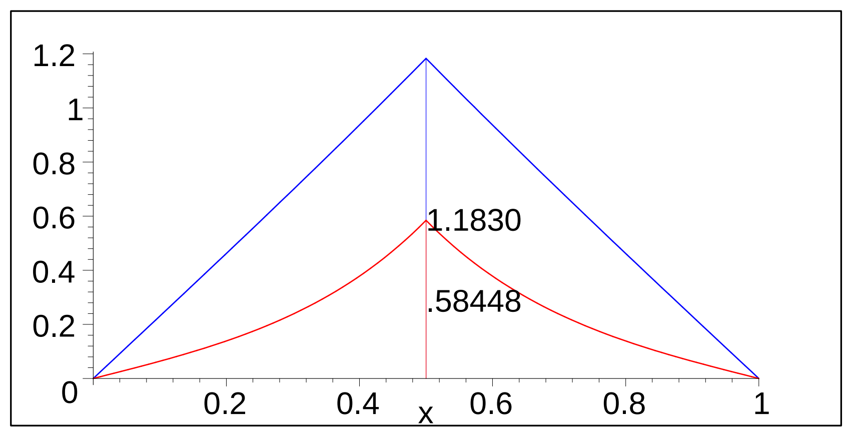
kąt skręcenia, φ [rad]



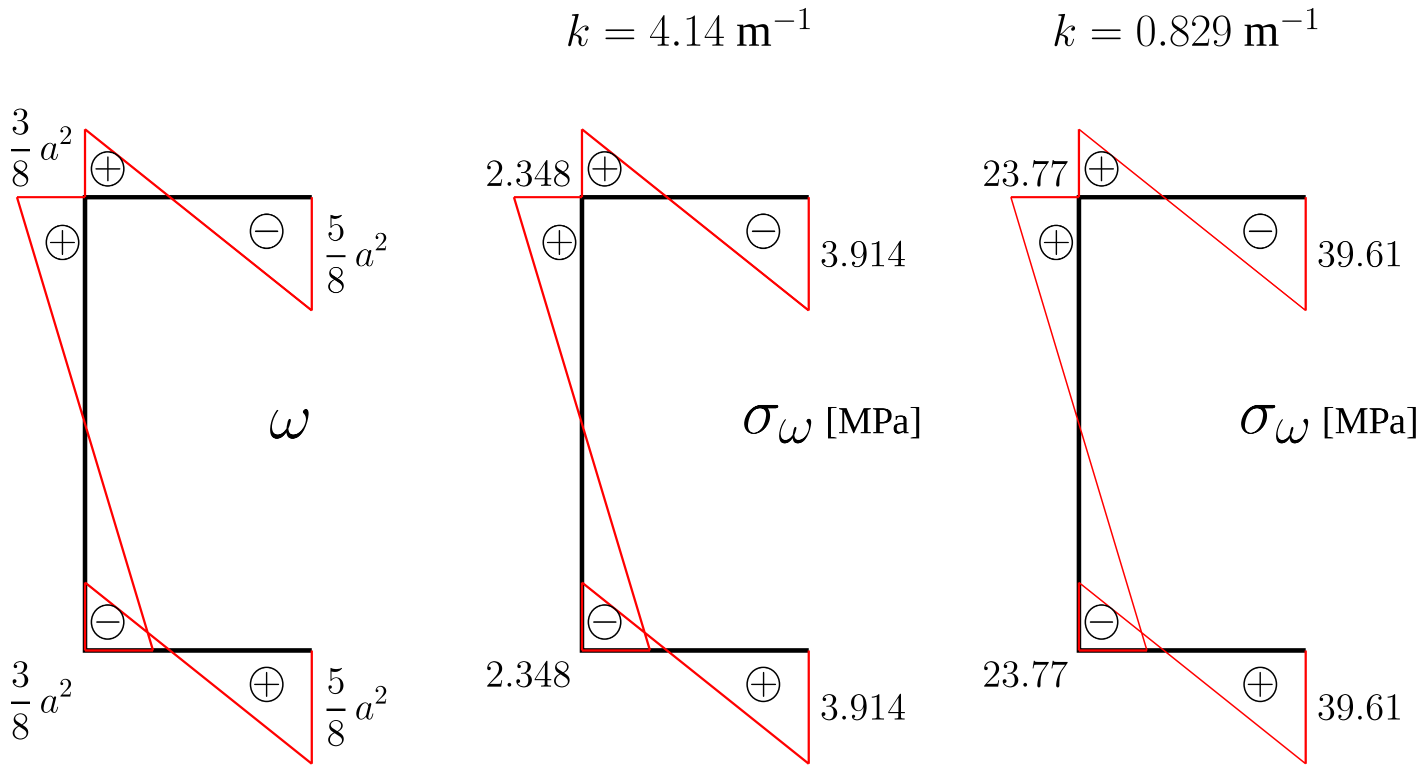
moment giętno–krętny, M_ω [Nm]



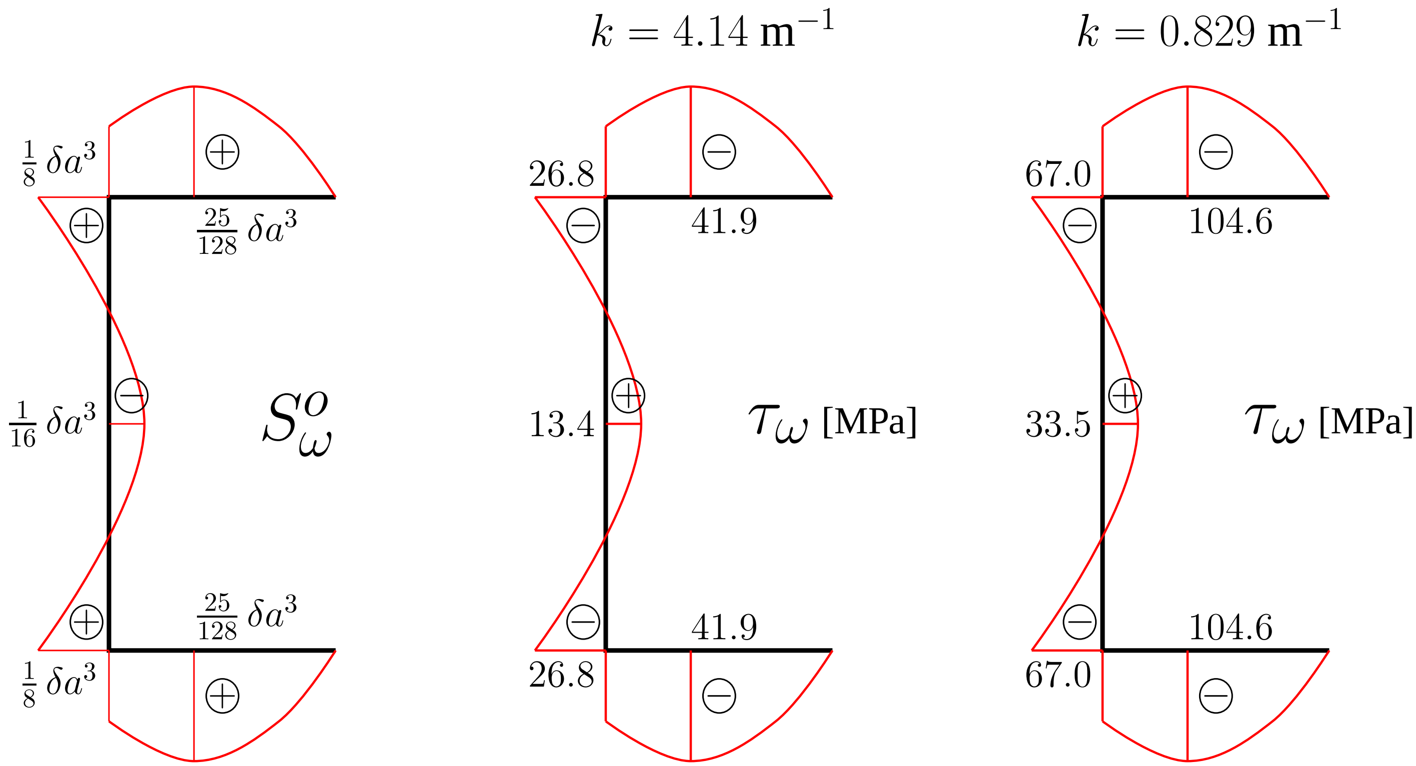
moment swobodnego skręcania, M_V [Nm]



bimoment, B_ω [Nm^2]

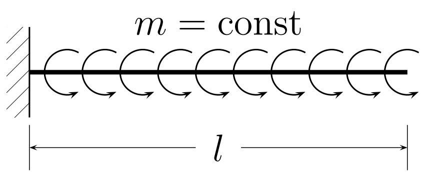


wycinkowe naprężenia normalne



wycinkowe naprężenia styczne

Przykład 2.



$m = 1$ N
$l = 1$ m
$\delta = 5,\ 1,\ 0.2$ mm

równanie różniczkowe bimomentu

$$B''_\omega - k^2\, B_\omega = -m$$

rozwiązanie

$$B_\omega(x) = A\, \mathrm{sh} kx + B\, \mathrm{ch} kx + \frac{m}{k^2}$$

pochodna funkcji bimomentu

$$B'_\omega(x) = Ak\, \mathrm{ch} kx + Bk\, \mathrm{sh} kx$$

warunki brzegowe

$$x = 0\colon\ \varphi' = 0 \Longrightarrow M_\omega + M_V = m\, l \Longrightarrow B'_\omega(0) = m\, l \qquad (1)$$

$$x = l\colon\ \sigma_\omega = 0 \Longrightarrow B_\omega(l) = 0 \qquad (2)$$

$$(1) \Longrightarrow A\, k = m\, l$$

$$(2) \Longrightarrow A\, \mathrm{sh} kl + B\, \mathrm{ch} kl + \frac{m}{k^2} = 0$$

stałe całkowania

$$A = \frac{m\,l}{k}$$
$$B = -\frac{m}{k^2\,\mathrm{ch}kl}\;(kl\,\mathrm{sh}kl + 1)$$

rozwiązanie

$$\begin{aligned} B_\omega &= \frac{m\,l}{k}\,\mathrm{sh}kx - \frac{m\,(kl\,\mathrm{sh}kl+1)}{k^2\,\mathrm{ch}kl}\,\mathrm{ch}kx + \frac{m}{k^2} \\ &= \frac{m}{k^2}\left(kl\,\mathrm{sh}kx - \frac{kl\,\mathrm{sh}kl+1}{\mathrm{ch}kl}\,\mathrm{ch}kx + 1\right) \end{aligned}$$

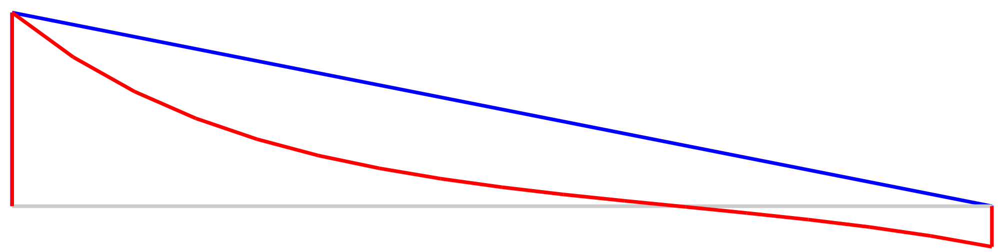
$$M_\omega \equiv B'_\omega = \frac{m}{k}\left(kl\,\mathrm{ch}kx - \frac{kl\,\mathrm{sh}kl+1}{\mathrm{ch}kl}\,\mathrm{sh}kx\right)$$

rozwiązanie dla pręta o bardzo cienkiej ściance

$$\lim_{k\to 0} B_\omega = -\frac{1}{2}\,m\,(l-x)^2$$
$$\lim_{k\to 0} M_\omega = -\,m\,(l-x)$$

$$k = 4.14\ \mathrm{m}^{-1}\ (\delta = 5\ \mathrm{mm})$$

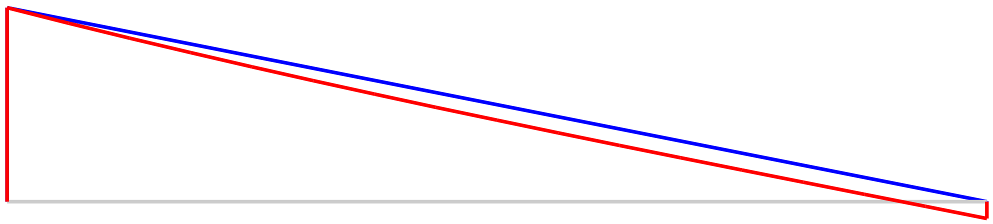
momenty M_ω, M_s ($\max = 1.000$ Nm)

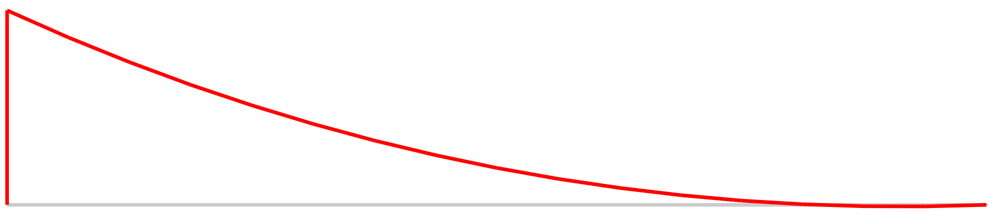


bimoment B_ω ($\max = 0.1848$ Nm^2)

$k = 0.829$ m^{-1} ($\delta = 1$ mm)

momenty M_ω, M_s ($\max = 1.000$ Nm)





bimoment B_ω ($\max = 0.4321$ Nm^2)

$k = 0.1658$ m^{-1} ($\delta = 0.2$ mm)

momenty M_ω, M_s ($\max = 1.000$ Nm)

bimoment B_ω ($\max = 0.4966$ Nm^2)